New Perspective on the Riemann Hypothesis

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Abstract

This tutorial provides a solid introduction to the Generalized Riemann Hypothesis and related functions, including Dirichlet series, Euler products, non-integer primes (Beurling primes), Dirichlet characters and Rademacher random multiplicative functions. The topic is usually explained in obscure jargon or inane generalities. To the contrary, this article will intrigue you with the beauty and power of this theory. The summary style is very compact, covering much more than traditionally taught in a first graduate course in analytic number theory. The choice of the topics is a little biased, with an emphasis on probabilistic models. My approach, discussing the "hole of the orbit" – called the eye of the Riemann zeta function in a previous article – is particularly intuitive.

The accompanying Python code covers a large class of interesting functions to allow you to perform as many different experiments as possible. If you are interested to know a lot more than the basics and possibly investigate this conjecture using machine learning techniques, this article is for you. The Python code also shows you how to produce beautiful videos of the various functions involved, in particular their orbits. This visual exploration shows that the Riemann zeta function (based on the trivial character χ), and a specific Dirichlet-L function (based on the non-trivial character χ_4), behave very uniquely and similarly, explaining the connection between the Riemann and the Generalized Riemann Hypothesis, in pictures and videos rather than words.

Contents

1 Introduction

Let $z = \sigma + it$ be a complex number: σ is the real part, and t is the imaginary part. Let P be a set of numbers, called primes: an element of P can not be factored into a product of elements of P. In some sense, numbers are to molecules what primes are to atoms. Typically but not always, P is the standard set of all prime numbers, or a subset of it, either finite or infinite. Finally p represents an element of P and $\chi(p)$ is any function taking on two possible values: $+1$ or -1 depending on p. The function $\chi(\cdot)$ is extended outside P using the formula $\chi(ab) = \chi(a)\chi(b)$, with $\chi(1) = 1$.

This article summarizes known properties and conjectures about various functions of z that can be represented by the following product, called Euler product:

$$\prod_{p \in P} \frac{1}{1 - \chi(p)p^{-z}}.$$

The most well known example is when the function $\chi(\cdot)$ is constant (thus equal to 1) and P is the full set of prime integers: this corresponds to the Riemann zeta function $\zeta(z)$. When expanded into a series, the Euler product becomes what is called a Dirichlet series. The series and product may not convergence on the same domain; when the series is conditionally but not absolutely convergent [Wiki] the product can diverge. This typically happens if $\sigma < 1$. It is the source of considerable difficulties, and the reason why the Generalized Riemann Hypothesis (GRH) is unproven to this day. Also, this explains why all the action takes place when $\frac{1}{2} \leq \sigma < 1$.

I won't discuss complex analysis in details here. It is sufficient to know that if $z = \sigma + it$, then $p^{-z} = \exp(-z \log p) = p^{-\sigma} \cos(t \log p) - ip^{-\sigma} \sin(t \log p)$. Also, the factors in the Euler product are ordered by increasing values of p. Without this specification, the product may be subject to multiple interpretations with different values, when convergence is conditional but not absolute. An introduction to the Riemann zeta and Dirichlet functions can be found in [4] and [21]. Finally, I occasionally use the term conditionally or absolutely convergent product. This intuitive concept is defined here.

1.1 Key concepts and terminology

The complex plane is the standard two-dimensional space: the real axis is the horizontal or X-axis; the imaginary axis is the vertical or Y-axis. The Riemann Hypothesis (RH) states that the Riemann zeta function $\zeta(z)$ defined earlier, has no root (that is, $\zeta(z) \neq 0$) if $\frac{1}{2} < \sigma < z$. I use the notation $\sigma = \Re(z)$ to indicate that σ is the real part of the complex number $z = \sigma + it$. Throughout this text, a positive number is a number ≥ 0. A number > 0 is called strictly positive.

The Generalized Riemann Hypothesis (GRH) makes the same statement as RH, for a larger class of well behaved functions, not just $\zeta(z)$. In short, it applies to Dirichlet functions where $\chi(\cdot)$ is completely multiplicative and periodic: these are called Dirichlet-L functions. Here, we limit ourselves to $\chi(p) \in \{-1, +1\}$. However, we also consider $\chi(\cdot)$'s that are not periodic. The classic non-trivial periodic $\chi(\cdot)$ is $\chi = \chi_4$, the non-trivial Dirichlet character modulo 4. It leads to an Euler product suspected to converge if $\sigma > \frac{1}{2}$. Proving the convergence, even only at $\sigma = 0.99$, would be a major milestone towards proving GRH. Of course, the product converges if $\sigma > 1$. If $\chi(\cdot)$ is allowed not to be periodic, there are known cases, discussed in this article, that meet the requirements of GRH. Typically these functions are much less interesting and do not satisfy a Dirichlet-like functional equation.

1.2 Orbits and holes

An original concept, the hole of the orbit of a Dirichlet function, is introduced in this article for the first time. It is epitomized in Figure 1 dealing with finite Euler products, and in Figure 5 featuring the orbit of truncated Dirichlet series (the truncated expansion of an infinite product, or in other words, the partial sums). In the RH and GRH contexts, the hole is present if $\frac{1}{2} < \sigma < 1$ is fixed and $0 < t < T$ is bounded. But as $T \to \infty$, the hole shrinks to a singleton at the origin in the absence of roots ($\frac{1}{2} < \sigma < 1$), or to an empty set if roots are present ($\sigma = \frac{1}{2}$). Studying modified Dirichlet functions that always have a hole may be key to making progress towards RH and GRH. In particular, it is interesting to study the behavior at their limit as they approach standard Dirichlet functions, and the hole slowly evaporates. This is the topic of sections 3.1 and 3.3.

The hole is a circle of maximum radius, with center on the X-axis (due to symmetry), that the orbit never crosses. The center of the hole may not be at the origin (see section 2.2.1 for examples), even when there is no root. The absence of root can be caused by a hole, or because the orbit never gets too close to the Y-axis. The orbit on a fixed domain $0 < t < T$, for a fixed σ, is defined as the set of all possible values of the corresponding Dirichlet function in the complex plane, as t (called the "time"), varies continuously between $t = 0$ and $t = T$.

Here, t is the imaginary part of the argument $z = \sigma + it$. The full orbit corresponds to $T = \infty$. The size of the hole as well as its presence/absence and location, depend on σ, and of course on T, P and $\chi(\cdot)$.

Finally, the hole is called a repulsion basin in the context of dynamical systems. The orbit may be bounded if $\sigma > 1$ or unbounded otherwise (in that case, typically extending to the entire complex plane).

1.3 Industrial Applications

While not discussed in this article, there are very interesting industrial applications of GRH. See my article on the prime test [9] used to test and design better pseudo-random number generators for cryptography purposes, based on Rademacher random multiplicative function and the Dirichlet character modulo 4. See also my article featuring synthetic data in machine learning applications [8], based on the orbits of Dirichlet functions and used to benchmark classification algorithms.

2 Euler products

I start with finite Euler products, where everything works fine: here the set P is a finite subset of the prime integers; there is no convergence issue, and orbits always have a hole. I first introduce the Dirichlet version $\eta(z)$ of the Riemann function $\zeta(z)$, as we need it to extend the convergence domain from $\sigma > 1$ to $\sigma > 0$ in order to study the behavior (presence of a hole and/or roots) when $0 < \sigma < 1$. For $\zeta(z)$, the function $\chi(\cdot)$ is constant and equal to 1. I then move to arbitrary $\chi(\cdot)$'s including Dirichlet characters modulo 4, and to infinite products. The function χ is extended to all integers via the formula $\chi(ab) = \chi(a)\chi(b)$: this extension leads to the fundamental Formulas (4), (5) and (6), linking the Euler product to its Dirichlet series expansion whenever both converge. The Euler product establishes the connection between the analytic properties of Dirichlet functions, and the distribution of prime numbers.

2.1 Finite Euler Products

Let $p_1 = 2$ and $P = \{p_1, p_2, \dots, p_d\}$ be a set of primes, listed in increasing order. Let $Q = \{q_1, q_2, q_3, \dots\}$ be the set of all $p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots$ where the a_i's are positive integers (including zero). The elements $q_1, q_2, \dots$ are also listed in increasing order. Thus $q_1 = 1$ and $q_2 = 2$.

The Dirichlet eta function $\eta_P(z)$ induced by P is then defined as

$$\eta_P(z) = \sum_{k=1}^{\infty} \delta_k q_k^{-z} = (1 - 2^{1-z}) \prod_{p \in P} \frac{1}{1 - p^{-z}}, \tag{1}$$

where $\delta_k = 1$ if q_k is odd, and $\delta_k = -1$ otherwise. The Euler product [Wiki] in formula (1) is finite and has d factors, while the infinite series always converges. Indeed, it can be proved (see Exercises 1) that

$$q_k \sim \exp\Big[k^{1/d}\Big(d! \prod_{p \in P} \log p\Big)^{1/d}\Big] \text{ as } k \to \infty. \tag{2}$$

If P is the set of all prime numbers, and thus $d = \infty$, then the product in formula (1) converges if $\sigma > 1$ while the alternating series [Wiki] converges if $\sigma > 0$. This can be proved using the Dirichlet test [Wiki], see here. For this reason, the series is called the analytic continuation of the product [Wiki]. This is the reason why we are interested in the alternating series, where $\delta_k \in \{-1, +1\}$, rather than in the series with $\delta_k = 1$: the latter, equal to the infinite product, diverges if $\sigma \leq 1$. This issue occurs only if $d = \infty$. See Exercise 3 for details. When the product diverges, the alternating series converges, but it is not absolutely convergent [Wiki].

Now let $\tau_z = |1 - 2^{1-z}|$. Assuming $z = \sigma + it$, the distance between $\eta_P(z)$ and the origin, is equal to

$$|\eta_P(z)| = \tau_z \prod_{p \in P} \frac{1}{|1 - p^{-z}|} = \tau_z \prod_{p \in P} \frac{1}{|1 - 2p^{-\sigma}\cos(t \log p) + p^{-2\sigma}|} > \tau_z \prod_{p \in P} \frac{1}{1 + p^{-\sigma}}. \tag{3}$$

Here $|\cdot|$ stands for the distance to the origin, and referred to as the modulus [Wiki] in complex analysis. An immediate consequence of Formula (3) is this: if $\sigma \neq 1$ is fixed and strictly positive, and if P only has a finite number of primes, then there is always a zone around the origin, with a strictly positive area, that the orbit of $\eta_P(z)$ will never hit or cross. This is illustrated in Figure 1. The zone in question corresponds to a hole in the orbit, also called repulsion basin [Wiki] in dynamical systems, for the red ($\sigma = 0.5$) and blue orbit ($\sigma = 0.75$). For the yellow orbit ($\sigma = 1.25$), the origin is outside the boundary of the orbit. The origin in Figure 1 is the black dot in the white area. Note that the "center" of the hole, in all three cases, is not the origin, but further to the right on the X-axis. See here the video corresponding to $P = \{2, 3, 5, 7\}$, and here for $P = \{2, 3, 5\}$.

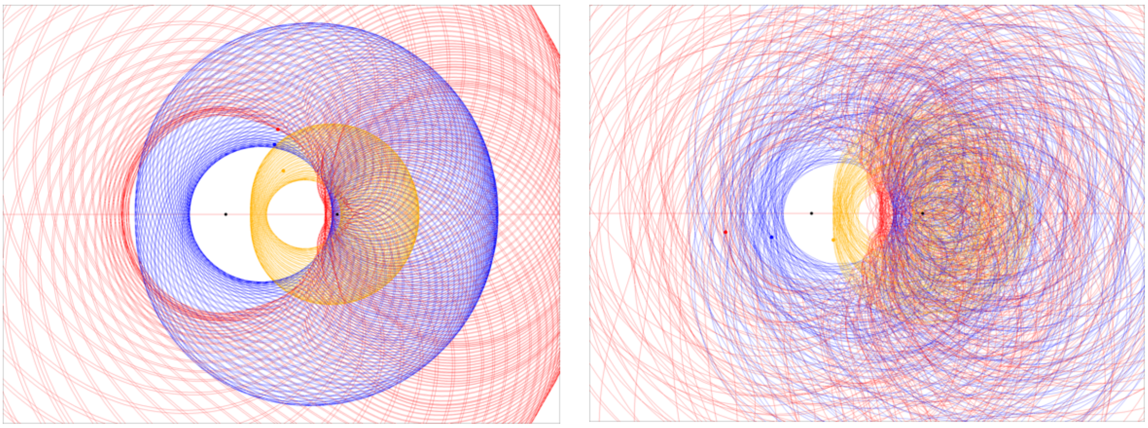

Figure 1: Three orbits ($\sigma = 0.5, 0.75, 1.25$) with finite Euler product: $P = \{2, 3\}$ (left) vs $\{2, 3, 5\}$ (right)

As you add more and more primes in P, the hole shrinks. In the end, when $d = \infty$, the hole shrinks to a single point (the origin) if we plot the whole orbit, rather the restricted orbit to $t \in [0, T]$ with $T = 1000$, as in Figure 1. Also, the center of the hole moves to the left on the X-axis (real axis) as σ is decreased from (say) 1.25 to 0.55.

For a fixed T, the center of the hole is denoted as z_0. It lies on the X-axis, and depends both on T and σ. It is tempting to scale the η function and replace it by $\sqrt{t} \cdot (\eta_P(z) - z_0)$, in the hope that when P is the full set of prime numbers and $T \to \infty$, the hole does not shrink to a singleton or an empty set. However, no matter how fast growing the scaling factor is (as a function of t), the universality property [Wiki] of the Dirichlet eta function implies that this goal can not be achieved. Generally speaking, working on finite Euler products and taking the limit $d \to \infty$, while very tempting, does not seem to lead to a proof of the Riemann Hypothesis, despite numerous attempts by many mathematicians including myself.

2.1.1 Generalization using Dirichlet characters

Formula (1) can be generalized as follows:

$$\eta_P(z, \chi) = \sum_{k=1}^{\infty} (-1)^{\Omega(q_k)} q_k^{-z} = (1 - 2^{1-z}) \prod_{p \in P} \frac{1}{1 - \chi(p) p^{-z}}, \tag{4}$$

where $\chi(p) \in \{-1, +1\}$. The product and the series may have different domains of convergence. However, on the domain where both converge, they have the same roots (except when $2^{1-z} = 1$, that is, when $\sigma = 1$ and $t = 2m\pi/\log 2$ with m an integer).

The generalized Omega function [Wiki] is defined as follows: if the prime factorization of q_k is $q_k = p_1^{a_1} p_2^{a_2} \cdots p_d^{a_d}$, then $\Omega(q_k) = a_1 \chi(p_1) + \cdots + a_d \chi(p_d)$. The function $(-1)^{\Omega(q_k)}$ is denoted as $\lambda(q_k)$ and referred to as the generalized Liouville function [Wiki]. The series in Formula (4) is called a Dirichlet-L function [Wiki]. The function χ can be extended to all strictly positive integers as follows: $\chi(1) = 1$, $\chi(p) = 0$ if $p \notin P$, and $\chi(ab) = \chi(a)\chi(b)$. Then $\chi(q_k)$=$\lambda(q_k)$ is a completely multiplicative function [Wiki].

From now on, all functions denoted as $\chi(\cdot)$ are assumed to be completely multiplicative, and thus uniquely characterized by the values they take on prime arguments. Now, let $L_P(z, \chi) = (1 - 2^{1-z})^{-1} \eta_P(z, \chi)$. Then we have:

$$L_P(z, \chi) = \sum_{k=1}^{\infty} \chi(k) k^{-z}, \quad \eta_P(z, \chi) = (1 - 2^{1-z}) L_P(z, \chi) = \sum_{k=1}^{\infty} (-1)^{k+1} \chi(k) k^{-z}, \tag{5}$$

$$L_P(z, \chi) = \prod_{p \in P} \frac{1}{1 - \chi(p) p^{-z}}, \quad |L_P(z, \chi)| = \prod_{p \in P} |1 - \chi(p) p^{-z}|^{-1} \geq \prod_{p \in P} \frac{1}{1 + p^{-\sigma}}. \tag{6}$$

If for a fixed integer $m > 1$, we have $\chi(p) = \chi(q)$ whenever $p, q \in P$ are two primes with $p \equiv q \bmod m$, and if in addition $\chi(p) = 0$ if p divides m, then $\chi(\cdot)$ is called a Dirichlet character modulo m [Wiki]. The standard Omega and Liouville functions correspond to the case where χ is constant and equal to $+1$, and P is the set of all prime numbers. The standard Liouville function also satisfies

$$\sum_{k=1}^{n} \lambda(k) \left\lfloor \frac{n}{k} \right\rfloor = \lfloor \sqrt{n} \rfloor, \quad L(n) = \sum_{k=1}^{n} \lambda(k) = \sum_{k=1}^{n} \mu(k) \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor,$$

where $\mu(k)$ is Liouville's sister function, called the Möbius function [Wiki]. See here for details. The brackets stand for the integer part function. The partial sums of the Liouville function is denoted as $L(n)$, while the partial sums of the Möbius function is the Mertens function [Wiki], and denoted as $M(n)$. If $q = q_k = p_1^{a_1} p_2^{a_2} \cdots p_d^{a_d}$, then $\mu(q_k) = (-1)^{w(q_k)}$, with $\omega(q_k) = \chi(p_1) + \cdots \chi(p_d)$. The standard Möbius function corresponds to the case where χ is a constant function equal to 1, and P is the set of all prime numbers.

Finally, Formula (3), providing a lower bound for the distance between the origin and any point on the orbit of $\eta_P(z, \chi)$, is still applicable and remains unchanged. In particular, if P is the set of all primes (or a big enough, infinite subset), the lower bound is zero due to divergence of the product. If that lower bound is indeed reached, even if asymptotically only, then the hope to find a hole bigger than a singleton evaporates.

2.2 Infinite Euler products

If P is the set of all prime numbers, the products in Formula (1), (3) and (4) become infinite. As a result, if $0.5 \leq \sigma < 1$, the holes in the orbit in Figure 1 may shrink to an empty set ($\sigma = 0.5$) or a single point – the origin – if $0.5 < \sigma < 1$. Actually, it may well be an empty set too in the latter case; nobody knows. But the Riemann hypothesis states that it should be a single point. This is the situation if χ is a constant function equal to 1. This function is called principal character in this context. But what happens if χ is not a constant? In this latter case, depending on χ, the orbit may never get too close to the origin.

Let us consider the completely multiplicative function χ called non-trivial Dirichlet character modulo 4, and denoted as χ_4 or $\chi_{4,1}$. It is uniquely characterized by its values on prime numbers p, as follows: $\chi_4(p) = +1$ if $p \bmod 4 = 1$, $\chi_4(p) = -1$ if $p \bmod 4 = 3$, and $\chi_4(2) = 0$. It satisfies $\chi_4(k+4) = \chi(k)$ for all positive integers. Again, P is the set of all primes.

Primes satisfying $p \bmod 4 = 3$ seem to be more numerous than the other ones, at least the smaller ones: they get a good head start. This is known as Chebyshev's bias [Wiki]. If these two types of primes are not evenly distributed, the orbit could get arbitrarily close to the origin. However, thanks to Dirichlet's theorem [Wiki], we know that the distribution is even. Thus one would expect that the Euler product in Formulas (4) and (6) would alternate nicely between $\chi_4(p) = +1$ and $\chi_4(p) = -1$ on average, thus converging for some $\sigma = \sigma_0$ smaller than one, and thus for all $\sigma > \sigma_0$. This is in contrast to the product in Formula (1), corresponding to the principal character $\chi(p) = 1$ for all p, denoted as $\chi_{4,0}$ and converging only for $\sigma > 1$.

Having no root if $\sigma_0 < \sigma < 1$ due to the non-vanishing product in Formula (4), one would conclude that the orbit of $L_P(z, \chi_4)$ never crosses the X-axis if $\sigma_0 < \sigma < 1$. However, to this day, nobody knows if the smallest possible value of σ_0, called the abscissa of absolute convergence [Wiki] is less than one. It is conjectured to be as low as 0.5, or lower. This is part of the Generalized Riemann Hypothesis [Wiki], an active research topic in number theory. Yet the associated series $L_P(z, \chi_4)$ defined in Formula (5) is conditionally convergent [Wiki] if $\sigma > 0$, see example 2.39 page 36, in Conrad [4].

For a reference focusing on completely multiple functions χ in the RH context, not just Dirichlet characters, see Borwein [1]. Finally, I discuss χ_4 in more details in section 3.2.3. In particular, I show the big contrast between $L_P(z, \chi_{4,1})$ and the standard Riemann zeta function $\zeta(z)$ corresponding to $L_P(z, \chi_{4,0})$.

2.2.1 Special products

Again, I investigate the infinite product $L_P(z, \chi)$ in Formula (6), with $z = \sigma + it$. If for some $0 < \sigma_0 < 1$, the set P contains infinitely many primes, but sufficiently spaced out so that

$$\rho = \prod_{p \in P} \frac{1}{1 + p^{-\sigma_0}} > 0,$$

then the orbit of $L_P(z, \chi)$ corresponding to $\sigma = \sigma_0$ will stay away from the origin, at a distance $\geq \rho$ at all times. This is true regardless of the function χ. In particular, it means that $L_P(z, \chi)$ has no root if $\sigma = \sigma_0$.

Now, if P is the set of all primes, $\chi(p) = 1$ for all primes (the standard case) and $\sigma = 0.5$, then $\eta_P(z, \chi)$, defined in Formula (4), has infinitely many roots. In addition, if its orbit gets too close to the origin, it gets attracted to it, otherwise it gets deflected: the origin seems to have an event horizon similar to that of a black hole. If you get too close, there is no way out, you will hit the origin very fast. See the blue curve in Figure 2, where the X-axis represents the time t, and the Y-axis the distance to the location $(c, 0)$ in the complex plane. For the blue curve, $c = 0$. Note that as long as $0.5 < \sigma < 1$ and $0 < t < T$ with T finite, the (finite) portion of the orbit exhibits a hole. But the hole shrinks very slowly to a single point, as $T \to \infty$. If $\sigma < 1$, the hole encompasses the origin at all times. But its actual center is located further to the right on the X-axis. See Figures 2, 3 and 4: each curve represents a distance to a specific location $(c, 0)$. The center moves to the left as T increases, or as σ decreases and gets closer to 0.5, until it merges with the origin. If $\sigma > 1$, the hole may not contain the origin, but the orbit is bounded and the origin is outside the external boundary of the orbit: see the yellow orbit in Figure 1.

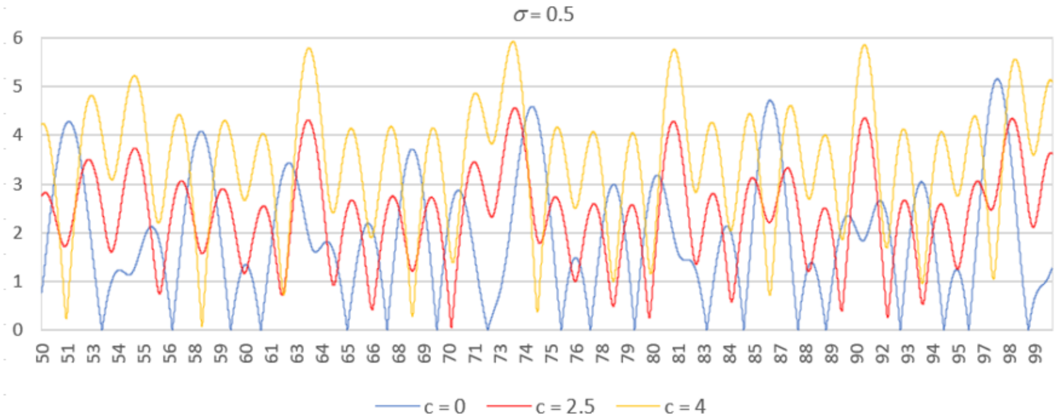

Figure 2: Distance between orbit and location $(c, 0)$ depending on t on the X-axis

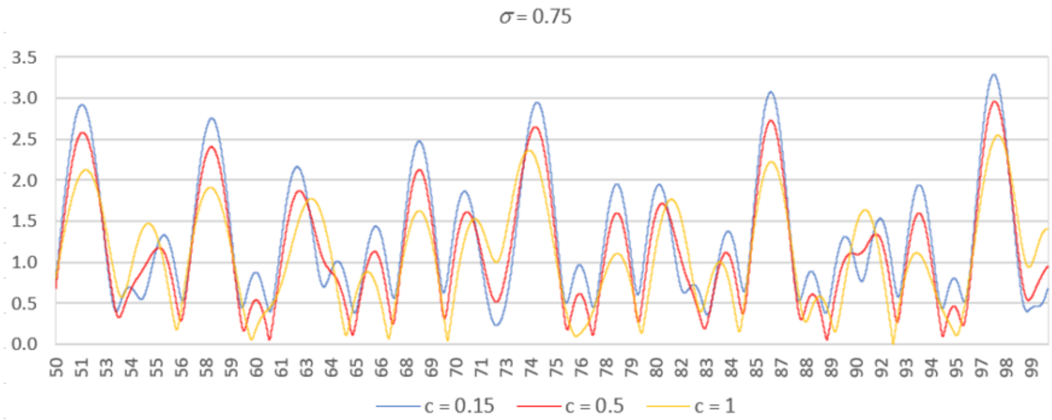

Figure 3: Distance between orbit and location $(c, 0)$ depending on t on the X-axis

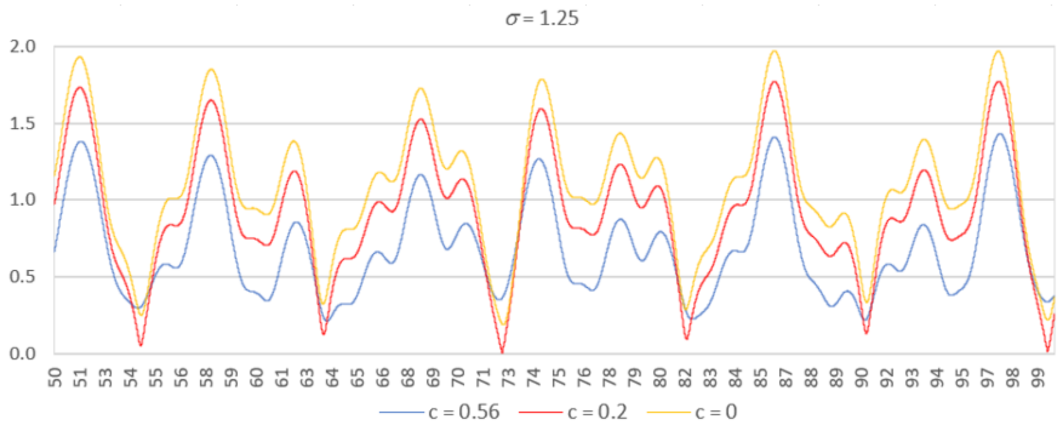

Figure 4: Distance between orbit and location $(c, 0)$ depending on t on the X-axis

Finally, another case worth investigating is as follows. Let P be the set of all primes, and p_k be the k-th prime with $p_1 = 2$. Define $\chi(p_{2k+1}) = +1$ and $\chi(p_{2k}) = -1$. Then the product in Formula (4) converges if $\sigma > 0.5$ (prove it for $t = 0$, using the fact that the product is alternating; see also here). Thus the infinite product (which has no root) can be used to compute $\eta_P(z, \chi)$. But the convergence is not absolute. I expect that there is no root if $0.5 < \sigma < 1$. But I also expect that the hole is reduced to a single point (the origin), as in the standard case.

2.2.2 Probabilistic properties and conjectures

Here P is the set of all primes, and $\chi(p) = 1$ for all p. The Liouville function $\lambda(\cdot)$ and Möbius functions $\mu(\cdot)$ introduced in section 2.1.1, have many interesting properties and open questions. Many are equivalent to or generalizing RH. Here I present here a brief summary; more can be found here. Again, let $L(n) = \lambda(1)+\cdots+\lambda(n)$.

- The numbers $\lambda(k)$, for positive integers k, are equal to -1 or $+1$ in equal proportions, thus averaging zero. But due to a good head start with negative values, it takes a long time for $L(n)$ to turn positive. In fact, the Pólya conjecture [Wiki] claims that $L(n)$ is always negative, but it was disproved in 1958. The smallest possible n satisfying $L(n) > 0$, namely $n = 906{,}180{,}359$, was found in 1980. The sign changes in $\lambda(k) = \pm 1$ occur somewhat randomly, as in independent Bernoulli trials. If the $+1$ and -1 were truly randomly distributed with zero mean, they would satisfy the law of the iterated logarithm [Wiki]:

$$\limsup_{n\to\infty} \frac{|L(n)|}{\sqrt{n \log\log n}} = C, \tag{7}$$

  for some constant C with $0 < C < \infty$. It is conjectured that this is not the case. In fact, the $\lambda(k)$'s can't possibly be independent, not even asymptotically, since the function is completely multiplicative. But to prove RH, all that is needed is a weaker statement, the fact $L(n)/\sqrt{n^{1+\epsilon}} \to 0$ as $n \to \infty$, for any $\epsilon > 0$. This is yet unproved. A similar conjecture exists for the Möbius function: the Mertens conjecture [Wiki], also implying RH, and thus yet unproved. See [14] for a stochastic version, based on random multiplicative Rademacher functions [Wiki], used as a substitute to emulate the "randomness" of the Möbius function.
- A stronger conjecture, yet not as strong as the law of the iterated logarithm, is this: the $\lambda(k)$'s behave like the binary digits of a normal number [Wiki]. In short, the number $\nu = \sum_{k=1}^{\infty} \lambda(k) 2^{-k}$ is normal. Of course ν is irrational, otherwise $\lambda(\cdot)$ would be a periodic function. It is not yet known if ν is transcendental, though some closely related numbers are [2]. The normality (or equivalently, ergodicity) of the sequence $\{\lambda(k)\}$ would imply that the Chowla conjecture, itself stronger than RH, is true: see [6]. But this is yet unproved.
- The numbers $\mu(k)$, for positive integers k, are equal to -1, 0, or $+1$. The proportion of those equal to zero is $1 - 6/\pi^2$. This is because $\mu(k) = 0$ if and only if k has a square factor. It is well known and easy to prove that the proportion of square-free integers [Wiki] is $6/\pi^2$. The proportions of $\mu(k)$'s equal to -1 or -1 are identical, a consequence of Dirichlet's theorem [Wiki].

Finally, an application of Kronecker's theorem [Wiki] leads to the following result: over time, for any fixed σ, the orbit of the Dirichlet L-functions defined by Formula (5) (assuming convergence) eventually fills a dense area in the complex plane. This is true whether the orbit is bounded or not, and whether it is has a "visible" hole or not. In other words, the image domain of $\eta_P(z, \chi)$ or $L_P(z, \chi)$ is a dense set [Wiki]. I provide an elegant proof of this fact in Exercise 5, using arguments similar to those used to prove its universality property [Wiki]. This implies that if $0.5 < \sigma < 1$, assuming P contains sufficiently many prime numbers, the Dirichlet eta function, regardless of χ, gets arbitrarily close to zero even though it may never actually hit zero. In other words, in that case, the hole eventually shrinks to a single point.

3 Finite Dirichlet series and generalizations

This section covers a large class of functions, starting with truncated modified Dirichlet series to assess the status of the hole in the orbit. I then move back to infinite Euler products in section 3.2, but this time not over the full set of primes as in section 2.2, but instead on infinite subsets arising from additive number theory. Some of these functions have no root if $\sigma > \sigma_0$, with (say) $\sigma_0 = 5/6$. They thus satisfy a weaker version of GRH, called quasi-GRH. Section 3.3 covers non-Dirichlet functions that don't have an Euler product, but behave like the Dirichlet eta function $\eta(z)$ with regard to the orbit and its hole. Here P is the set of all primes, but the sine and cosine attached to $p^{-z} = \sigma^{-z}\cos(t \log p) - i\sigma^{-z}\sin(t\log p)$ are now replaced by wavelets. Section 3.4 deals with non-integer primes (even matrices) that mimic the behavior of prime integers, and called Beurling primes. They come with Euler products too, and the corresponding Dirichlet series is now called a Dedekind zeta (or eta) function. Section 3.5 deals with random Dirichlet functions: their interest lies in the fact that the corresponding (random) Euler products converge, albeit conditionally, when $\frac{1}{2} < \sigma \le 1$. Thus they satisfy a probabilistic version of GRH, in particular the absence of root if $\sigma > \frac{1}{2}$.

3.1 Finite Dirichlet series

Formula (1) features a finite (Euler) product. However the corresponding series, on the left hand side, is infinite. Here I discuss a different approach, using a finite version containing the first n terms of the full Dirichlet series when P is the set of all prime numbers. The new function is defined as

$$\eta(z, \beta, n) = \sum_{k=1}^{n} \delta_k \lambda_k p_k^{-z}, \tag{8}$$

where $\lambda_k = 1/k$ for all k except $k = 2$. The coefficient λ_2 is denoted as β. Here, $\delta_k = 1$ if k is odd, otherwise $\delta_k = -1$. If n is infinite and $\beta = 1/2$, then $\eta(z, \beta, n)$ coincides with the standard Dirichlet eta function.

Despite the finite number of terms in the series, this approach is considerably more difficult. Unlike in section 2.1, there is no simple product (finite or infinite), to represent the truncated η function. While the approach in section 2.1 has a strong number theory flavor, here we are dealing with approximations and numerical analysis. Yet, the case $n = 3$ is trivial: the orbit fills a ring. The hole is a circle centered at $(1, 0)$, not at the origin. Both the interior and exterior boundaries of the orbit are circles, with known radius. See Exercise 4 for a complete solution.

But the general case is much more complicated. Of course the series corresponding to the finite Euler product results in an orbit with a solid hole of radius > 0, always encompassing the origin, and not shrinking to a singleton when you display the full, infinite orbit. But that series always has an infinite number of terms. As you increase the number of terms in the truncated series, keeping the first n terms only and $\beta = 1/2$, the hole disappears when $n \geq 5$ is small enough, only to reappear when $n > 50$. The hole then stays there all the way to $n = \infty$. As n increases, it eventually shifts to the left on the X-axis, to encompass the origin. For instance, at $n = 2000$, the origin is inside the hole. This is based on visually inspecting the orbit at $\sigma = 0.90$, with $0 < t < 2000$: see Figure 6.

For small values of n with no hole, increasing β is one way to re-introduce the hole in the orbit, as pictured in Figure 5. This leads to a possible new path to explore RH: using a large n, with $\beta > 1/2$ (the larger β, the larger the hole), and let $\beta \to 1/2$ as $n \to \infty$. In Exercise 4, I discuss some conditions that guarantee the presence of a hole encompassing the origin, for small values of n.

The Python code in section 5.1 allows you to replicate my experiments. You need to set the parameter `method` to `Eta`, and the parameter `Dirichlet` to `True`. This allows you to work with a series that converges when $\frac{1}{2} \leq \sigma < 1$, as opposed to $\sigma > 1$. Of course, for finite Euler products, the infinite Dirichlet series always converges regardless of σ, with or without `Dirichlet` set to `True`.

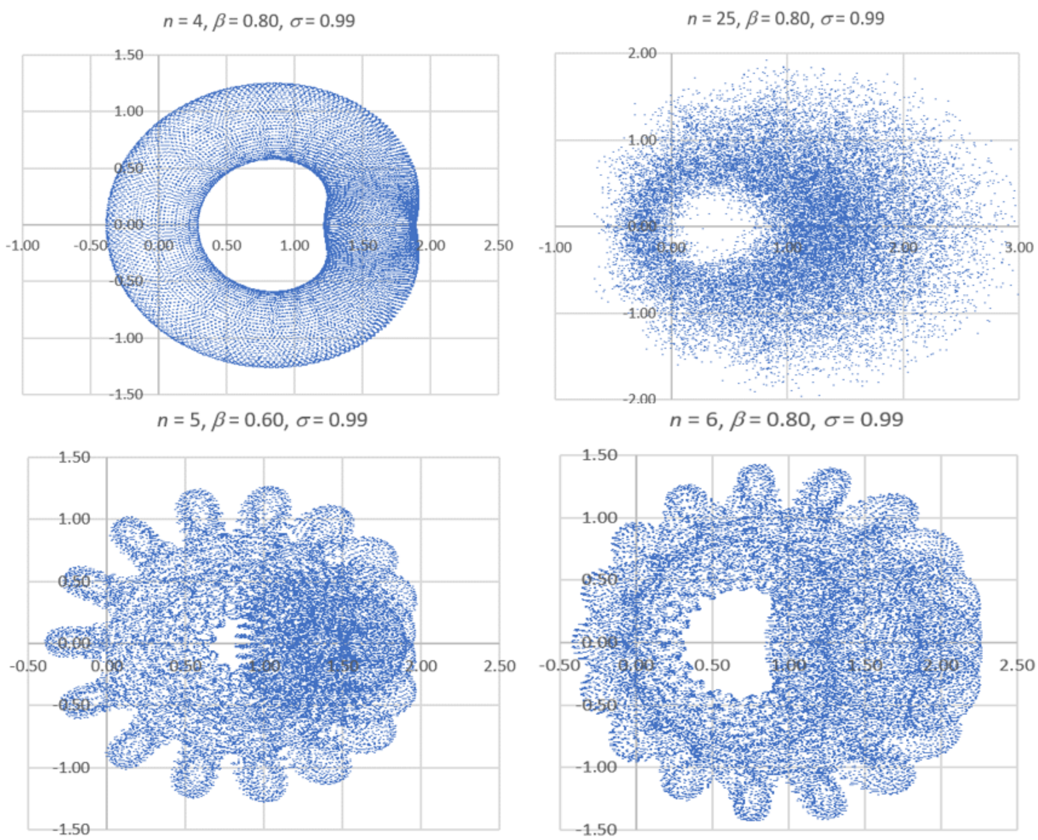

Figure 5: Four orbits where the "hole" (repulsion basin) is apparent

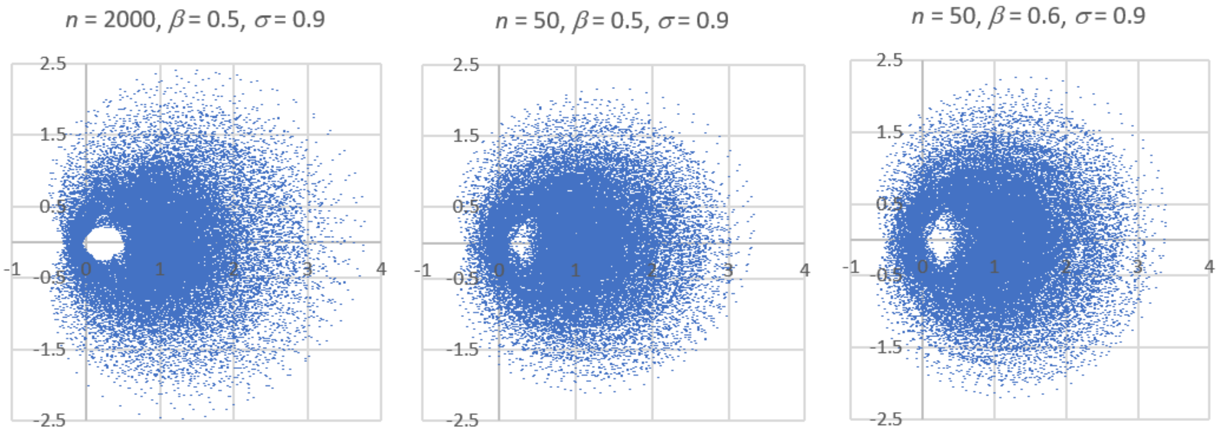

Figure 6: Three orbits with "hole" closer to the origin, showing impact of $\beta > \frac{1}{2}$ and larger n

3.2 Non-trivial cases with infinitely many primes and a hole

Now, let's get back to Euler products. Thus the associated Dirichlet series (the expansion of the product) always contains infinitely many terms, even if the product is finite. For a given σ, in order to get a hole encompassing the origin, with the distance between the orbit and the origin always $\geq \rho$ at all times t, the following must be satisfied:

$$\rho = \prod_{p \in P} \frac{1}{1 + p^{-\sigma}} > 0. \tag{9}$$

This is briefly discussed at the beginning of section 2.2.1. Note that this requirement is independent of the function χ. It is satisfied if the product is finite (that is, if the set P is finite) , or if $\sigma > 1$, or if $\sigma < 1$ and P is infinite but sparse enough so that the product converges. This section discusses the latter.

Because of the generalized universality property [Wiki] discussed in Exercise 5, the inequality $\geq \rho$ becomes an equality: the largest circle encompassing the origin, not crossed by the orbit, has radius exactly equal to ρ.

We already know that if P is the full set of primes and $\sigma < 1$, there is no hole. According to the Riemann Hypothesis, the hole at the origin is reduced to a singleton if $0.5 < \sigma < 1$, and to an empty set if $\sigma = 0.5$. The Generalized Riemann Hypothesis [Wiki] claims that this is true regardless of the Dirichlet character $\chi(\cdot)$ [Wiki]. But what if we use the set of twin primes [Wiki] for P? Then the product in Formula (9) converges if $\sigma = 1$, thus there is a hole with $\rho > 0$ encompassing the origin, if $\sigma = 1$. The convergence is a direct consequence of Brun's theorem [Wiki]: the fact that the sum of the inverse of twin primes converges. I did not investigate the case $\sigma < 1$.

I now discuss two cases that are known to satisfy (9). Both illustrate methods of additive number theory [Wiki]. The idea is to look at two (or more) sets of integers A and B, and then check the density of integers in $A + B$. The most well-known example is Goldbach's conjecture [Wiki]: it states that if $A = B$ is the set of all primes, then $A + B$, defined as the set of all integers $a + b$ with $a \in A, b \in B$, covers all positive even integers greater than 2. The second most well-known example is when $A = B$ is the set of square integers. The problem is referred to as sums of squares [Wiki]. The resulting $A + B$ is too large to be of interest here. A more general version is the sum of higher powers, known as Waring's problem [Wiki]. If $A = B$ is the set of positive cubes, or A is the set of positive cubes and B the set of squares, then $A + B$ is small enough to lead to interesting results. Let's now investigate these popular cases.

3.2.1 Sums of two cubes, or cuban primes

Let P be the set of primes, greater than 2, that are the sum of two cubes. They are called cuban primes [Wiki], and featured in the OEIS list of integer sequences as entries A334520 and A002407. These primes are of the form $3x^2 + 3x + 1$ where x is a positive integer. Because cuban primes are less numerous than square integers, we have $\prod_{p \in P}(1 + p^{-\sigma})^{-1} < \infty$ if $\sigma > 0.5$. Thus, if $\sigma > 0.5$ and regardless of χ, the associated $L_P(z, \chi)$ orbit has a hole at the origin, with radius > 0 (not a single point or an empty set). In short, $L_P(z, \chi)$ never gets too close to zero if $\sigma > 0.5$. This is in stark contrast to the standard Riemann zeta function $\zeta(z)$, which has a hole of strictly positive radius only if $\sigma > 1$.

It is conjectured that there are infinitely many cuban primes. For primes of the form $x^3 + 2y^3$, a proof was published in 2001, see [15].

3.2.2 Primes associated to elliptic curves

Now, let $P = \{p_1, p_2, \dots\}$ be the set of primes of the form $x^3 + y^2$, listed in increasing order. These primes are far less numerous than primes of the form $x^2 + y^2$, but far more numerous than primes of the form $x^3 + y^3$ (the cuban primes). Here x, y are positive integers. Note that if $p \in P$, then $y^2 = x^3 + p$ for some integers $x \leq 0, y \geq 0$. This is the equation of an elliptic curve [Wiki].

When n is large enough, the number of positive integers smaller than n, of the form $x^3 + y^2$, is less than $n^{5/6}$. We don't know how many of them are prime numbers, but we know that it must be less than that. Thus p_k is asymptotically larger than $k^{6/5}$. For details, see, Exercise 6, based on a general summation formula posted here. Then, using Formula (6), we have

$$\rho = |L_P(z,\chi)| \geq \prod_{k=1}^{\infty} \frac{1}{1 + p_k^{\sigma}} \geq \prod_{k=1}^{\infty} \frac{1}{1 + k^{6\sigma/5}}. \tag{10}$$

Regardless of the function $\chi(\cdot)$, the rightmost product in Formula (10) converges (absolutely) if $6\sigma/5 > 1$, that is, if $\sigma > \frac{5}{6}$. So, not only $L_P(z,\chi)$ has no zero if $\sigma > \frac{5}{6}$, but there is a circle of radius $\rho > 0$ centered at the origin (a hole), that the orbit never crosses.

For other Dirichlet-L functions with known abscissa of convergence [Wiki] $\sigma < 1$, see the article "Modular Elliptic Curves", pages 14–18, in [19]. Interestingly, (conditional) convergence is proved also for $\sigma > \frac{5}{6}$, by looking at the series rather than the product. Primes of the form $x^3 + y^2$ are listed in the Encyclopedia of Integer Sequences, as entry A066649. Related entries include A022549, A055393, A173795, and A123364.

Note: The theory of elliptic curves is now a hot topic in number theory. They were used in the proof of Fermat's last theorem [Wiki]. The proof was published in 1995, more than 350 years after it was first conjectured. The theorem states that $x^n + y^n = z^n$ has no non-trivial solution in integer numbers if $n > 2$.

3.2.3 Analytic continuation, convergence, and functional equation

If the Euler product converges only for $\sigma > 1$, you need to find and extension to $\sigma > 0.5$ to assess whether $L_P(z,\chi)$ has roots when $0.5 < \sigma < 1$. One way to do it to get an analytic continuation [Wiki], at least down to $\sigma = 0.5$. If $\chi(\cdot)$ is constant and equal to 1, try using the alternating Dirichlet series $\eta_P(z,\chi)$ defined by Formula (4), rather than the Euler product, to compute $L_P(z,\chi)$. It may converge over a larger domain. If the analytic continuation satisfies a standard Dirichlet functional equation [Wiki] and $z_0 = \sigma + it$ is a root with $0 < \sigma < 1$, then $1 - z_0$ is also a root. Thus in that case, if there is no root with $0.5 < \sigma < 1$, then any possible root with $0 < \sigma < 1$ must have $\sigma = 0.5$. The functional equation, when it exists, can be derived using exponential sums such as $\sum_{k=1}^{\infty} \exp(-\pi k^2 y)$. See section 4.1 (page 71) in Conrad [4].

If for some σ_0 the Dirichlet series converges, then it converges for all z with $\Re(z) = \sigma > \sigma_0$. The abscissa of conditional convergence for the η_P series defined in Formula (5), is denoted as σ_c. It is the minimum value satisfying

$$\sum_{k=1}^{\infty} (-1)^{k+1} \chi(k) k^{-\sigma_c} < \infty. \tag{11}$$

The abscissa of absolute convergence, denoted as σ_a, is the minimum value satisfying

$$\sum_{k=1}^{\infty} |\chi(k)| k^{-\sigma_a} < \infty.$$

Note that $\chi(k) \in \{-1, 0, +1\}$. It is equal to 0 only if k can not be expressed as a product of elements of P. This happens when P is not the full set of primes. Similar arguments can be used to obtain the abscissa of convergence for the L_P series, or to study the convergence of the product. For instance, for the product in Formula (4), conditional convergence is equivalent to the convergence of the series $\sum_{p \in P} \chi(p) p^{-\sigma}$. In particular, for $\chi = \chi_4$, the Dirichlet character modulo 4 introduced at the beginning of section 2.2, the series for $L_P(z,\chi)$ has $\sigma_c = 0$ because χ_4 is periodic, and thus the series in Formula (11) is alternating.

3.2.4 Hybrid Dirichlet-Taylor series

An interesting generalization of the Euler product, with $\chi(p) = x^{\nu(p)}$, is as follows:

$$L_P(z, x, \nu) = \prod_{p \in P} \frac{1}{1 - x^{\nu(p)} p^{-z}} = \sum_{k=1}^{\infty} \varphi(k) x^{\Omega_\nu(k)} k^{-z},$$

where $\nu(p)$ is defined on the primes $p \in P$. If the unique factorization of k, using primes in P, is $k = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots$ then $\Omega_\nu(k) = a_1 \nu(p_1) + a_2 \nu(p_2) + \dots$, where the latter sum is actually finite. Here the a_i's are integers ≥ 0.

If k can not be factored in P, for instance if P does not contain all the prime integers, then $\varphi(k) = \Omega_\nu(k) = 0$ otherwise $\varphi(k) = 1$. Note that $\Omega_\nu(\cdot)$ is a function defined on $Q = Q_P$, the multiplicative group generated by P, containing all product combinations of elements of P (including 1). In fact, $\Omega_\nu(\cdot)$ generalizes the Omega function [Wiki]. If the elements of Q_P are denoted (in increasing order) as q_1, q_2 and so on, then the following notation is more flexible:

$$L_P(z, x, \nu) = \prod_{p \in P} \frac{1}{1 - x^{\nu(p)} p^{-z}} = \sum_{k=1}^{\infty} x^{\Omega_\nu(q_k)} q_k^{-z}. \tag{12}$$

This notation works even if the p_k's (and thus the q_k's) are not integers, as in section 3.4. We can also define $\eta_P(z, x, \nu)$ using the same mechanism as in Formula (5); it may provide an analytic continuation when $x \to 1$.

Examples

Assuming P is the set of all prime integers, interesting examples include:

- If $\nu(p) = 1$ and $x = 1$, then $L_P(z, x, \nu) = \zeta(z)$, the Riemann zeta function [Wiki]. If $\nu(p) = 1$ and $x = -1$, then $L_P(z, x, \nu) = \zeta(2z)/\zeta(z)$. If $\nu(p) = \log p$ and $0 < x \leq 1$, then $L_P(z, x, \nu) = \zeta(z - \log x)$. If $\nu(p) = p$ and $-1 < x < 1$, then we have absolute convergence when $\sigma \geq 0$. In addition the series in Formula (12) is a Taylor series in x, and a Dirichlet series in z. This function has no roots, but it does have poles [Wiki]. Furthermore, $\lim_{x \to 1} L_P(z, x, \nu) = \zeta(z)$.
- Let $x = -1$ and $\nu(p) = 2d(\pi(p), \alpha) - 1$, where $\pi(\cdot)$ is the prime-counting function [Wiki] and $d(k, \alpha)$ is the k-th binary digit of the real number $0 < \alpha < 1$. Choose α so that its binary digits are random enough, behaving like an infinite fair coin-tossing game. Then, by virtue of the Glivenko-Cantelli theorem [Wiki], the empirical distribution [Wiki] of the digits converge to the underlying theoretical distribution of the process described in section 3.5. In particular, we have both convergence of the product and no root if $\sigma > 0.5$. Thus the Generalized Riemann Hypothesis [Wiki] (abbreviated as GRH) is verified in this case. You still need to find some α that works, for instance some normal number [Wiki] that would fit the bill. Of course $\alpha = \sqrt{2}/2$ is a great candidate, but no one knows if it is normal or not, depite the fact that it successfully passed all the statistical tests ever designed.
- In the previous example, if $\alpha = 2/3$, GRH is also satisfied, despite the lack of randomness: the digits alternate perfectly between 0 and 1. But $p_{2k}^{-\sigma} - p_{2k+1}^{-\sigma} \to 0$ fast enough as $k \to \infty$ (see here), thus we have convergence of the product if $\sigma > 0.5$ and therefore, no root. Now, let α be defined as follows: the first digit is zero; then $d(\pi(p), \alpha) = 1$ if $p \equiv 1 \bmod 4$ and $d(\pi(p), \alpha) = 0$ if $p \equiv 3 \bmod 4$. Then $L_P(z, x, \nu) = L(z, \chi_4)$ where χ_4 is the non-trivial Dircichlet character modulo 4. The digits have limited randomness; the proportion of zero/one is 50/50 thanks to Dirichlet's theorem [Wiki], and this Dirichlet-L function [Wiki] enjoys a number of interesting properties. If its product converges when $\sigma > 0.5$ (nobody knows), then $L(z, \chi_4)$ would also satisfy GRH. The non-randomness of the digits (this in itself does not rule out GRH) is caused by the fact that $\chi_4(\cdot)$ is completely multiplicative and periodic. In particular, if p is a prime, then $\chi_4(p^2) = 1$ and thus $d(\pi(p^2), \alpha) = 1$.
- Let $\nu(p) = 1/p$ and $0 < x \leq 1$. When $\nu(p) = \log p$, you approach ζ (when $x \to 1$) in the exact same way as moving to the left on the real axis, from $\sigma > 0.5$ (no root if GRH is true) to $\sigma = 0.5$ (infinitely many roots). The case $\nu(p) = 1/p$ offers a different perspective. Also $x^{1/p} \to 1$ as $p \to \infty$, regardless of $0 < x < 1$. Thus $\nu(p) = 1/p$ is appealing. For instance, when $x = 0.99$ and $\sigma = 0.55$, the orbit of $\eta_P(z, x, \nu)$ has a small hole around the origin (if might shrink to a singleton if you display the full orbit for all $t > 0$, as it does if $x = 1$). The orbit regularly gets close to the origin before moving away again: this happens only when t is close to the imaginary part of a non-trivial root of ζ. For a fixed value of σ (say 0.5), increasing x from (say) 0.4 to 0.8 will move the hole to the left, closer to the origin as expected. Of course, $\lim_{x \to 1} L_P(z, x, \nu) = \zeta(z)$. For instance, let $\sigma = \frac{1}{2}$. Then if $x = 0.4$ the orbit has a massive hole centered around $(0, 1)$ on the X-axis. If $x = 0.8$, the orbit has a massive hole centered around $(0, \frac{1}{2})$.

3.3 Riemann Hypothesis with cosines replaced by wavelets

The standard Dirichlet eta function $\eta(z)$, with $z = \sigma + it$ can be represented by two non-periodic trigonometric series: one for the real part involving cosine terms, and one for the imaginary part (a phase shift of the first one) involving sine terms. If you modify even very slightly the coefficients in these series, you lose the interesting properties: Dirichlet functional equation [Wiki], infinite number of roots when $\sigma = \frac{1}{2}$ (in other words, the orbit passing through the origin over and over), no root and hole in the orbit when $\frac{1}{2} < \sigma < 1$.

But what about a drastic change, replacing the sine and cosine functions by other periodic functions? Depending on the replacement, this actually works (except for the functional equation and therefore the roots when $\sigma = \frac{1}{2}$), proving that there is nothing special about the sines and cosines when dealing with the Riemann

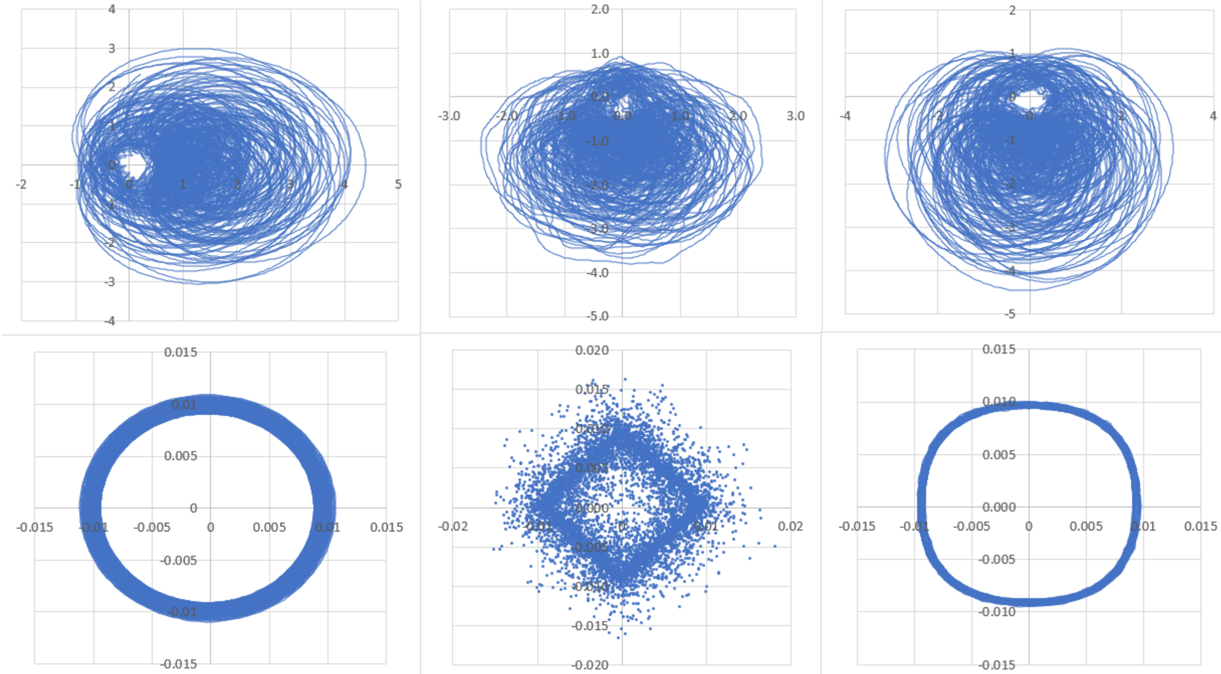

Figure 7: Orbit of Dirichlet eta $\eta(z)$ when cosines are replaced by other periodic functions

Hypothesis, at least when $\frac{1}{2} < \sigma < 1$. I start by introducing the following real-valued function:

$$\varphi(z, \theta) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{W(\theta + t \log k)}{k^{\sigma}}, \tag{13}$$

where $z = \sigma + it$, $0 \leq \theta < 2\pi$ and W is a periodic function of period 2π, to be defined later. I also use the notation $\varphi_1(z) = \varphi(z, 0)$ and $\varphi_2(z) = \varphi(z, \pi/2)$. In particular, if $W(x) = \cos x$, then $\varphi_1(z)$ is the real part of $\eta(z)$, and $\varphi_2(z)$ is the imaginary part. Also, here P is the full set of prime integers and $\chi(\cdot)$ is the constant function equal to 1.

The top part of Figure 7 shows the orbit of the modified $\eta(z)$ function for $\sigma = 0.75$ and $0 < t < 200$, using the three functions W discussed in this section. The left plot corresponds to the cosine wave (in this case $\eta(z)$ is the standard Dirichlet function), the middle plot to the triangular wave, and the right plot to the alternating quadratic wave.

- Triangular wave:

$$W(x) = \begin{cases} -2x/\pi, & \text{if } 0 \leq x \leq \pi/2 \\ -2 + 2x/\pi, & \text{if } \pi/2 \leq x \leq 3\pi/2 \\ 4 - 2x/\pi & \text{if } 3\pi/2 \leq x \leq 2\pi \end{cases}$$

- Alternating quadratic wave:

$$W(x) = \begin{cases} -4x(x-\pi)/\pi^2, & \text{if } 0 \leq x \leq \pi \\ 4(x-\pi)(x-2\pi)/\pi^2 & \text{if } \pi \leq x \leq 2\pi \end{cases}$$

- Cosine wave:

$$W(x) = \cos x.$$

I used the first 20,000 terms in the Formula (13). In each case, the origin is inside the hole, clearly showing the absence of roots when $0 < t < 200$. Other waves tested do not exhibit a hole. The bottom plots show the corresponding errors, magnified by a factor 20: it represents the difference between the approximate computations based on 2000 terms, and the more accurate results based on 20,000 terms.

3.4 Riemann Hypothesis for Beurling primes

Beurling primes are real or complex numbers used to mimic and study distributions related to prime integers. A Beurling prime set $P = \{p_1, p_2, \dots\}$ is any set of real or complex numbers with the constraint that the product of elements of P can not be an element of P. We then define $Q = Q_P$ as the set of all product combinations

$q = p_1^{a_1} p_2^{a_2} \cdots$ where $a_1, a_2, \dots$ are integers ≥ 0. It is also required that the factorization of $q \in Q_P$, using Beurling primes from P, is unique.

A counter example is the set of Hilbert primes [Wiki]: for instance, $1617 = 21 \times 77 = 33 \times 49$ where $21, 33, 49, 77$ are Hilbert primes (they can not be factored as a product of Hilbert primes). A good example is when Q is the set of numbers that are the sum of two square integers: if $q, q' \in Q$ then $q \cdot q' \in Q$. In this case, $P = \{2, 5, 9, 13, \dots\}$, see here. Note that $9, 49$ and 121 are primes in Q. Related to this set is the set of Gaussian primes [Wiki], which are complex numbers.

The Python code in section 5.1 handles Beurling primes. In the current implementation, P is the set of all prime integers except that 3 is replaced by $2 + \log 3 \approx 3.0986$. When using the Beurling option, set the `Dirichlet` parameter to `True`, to get convergence when $\frac{1}{2} \leq \sigma < 1$. The orbit of $\eta_P(z, \chi)$, assuming $\chi(\cdot)$ is constant and equal to 1, exhibits a hole that encompasses the origin if $\frac{1}{2} < \sigma < 1$. The hole shrinks to a point (the origin) if the full, infinite orbit is plotted, pointing to the absence of roots, just like for the standard Dirichlet eta function $\eta(z)$. Of course this is part of the GRH conjecture, not a proven fact. Likewise, if $\sigma = \frac{1}{2}$, there are infinitely many roots. In the context of Beurling numbers, the associated Dirichlet function $L_P(z, \chi)$ is called a Dedekind zeta function [Wiki], and Dedekind eta for the alternating series $\eta_P(z, \chi)$.

Beurling primes can be defined for objects other than numbers, like polynomials or matrices. For instance, let A be a square matrix, and define $p_k = \exp(\mu_k A)$, where the μ_k's are distinct, strictly positive real numbers ordered by increasing values, and linearly independent over the set of positive integers, so that the factorization in Q_P is unique. For instance, μ_k is the logarithm of the k-th prime integer. Any element (matrix) $q \in Q_P$ can be written as

$$q = p_1^{a_1} p_2^{a_2} \cdots = \exp(|q|A) = \sum_{k=0}^{\infty} |q|^k \frac{A^k}{k!}, \quad \text{with } |q| = \sum_{k=1}^{\infty} a_k \mu_k.$$

Here $a_1, a_2, \dots$ are integers ≥ 0 and $|q|$ is called the norm. We can define an order on Q as follows: if $q, q' \in Q$, then $q < q'$ if and only if $|q| < |q'|$. Note that q, q' and the p_k's are matrices. We can build Dirichlet series and Euler products for these primes, and study properties when $\frac{1}{2} \leq \sigma < 1$, as we do for standard prime integers. For more on Beurling primes, see [5] and [16]

3.5 Stochastic Euler products

There are different ways to randomize functions related to Euler products. You may want to randomize $L_P(z, \chi)$ or $\eta_P(z, \chi)$ using complex random variables [Wiki]. Or you may want to randomize the real or imaginary parts of these functions, or their norm. These random products have gained a lot of interest recently, at they provide insights about RH and its generalized version, GRH. For a recent reference on random Euler products, see [18]. I briefly discussed randomized multiplicative functions such as random Rademacher functions [Wiki] in section 2.2.2. More on this topic can be found in [9, 12, 13, 14, 18]. My recent book on stochastic processes [11] discusses tiny random perturbations applied to Dirichlet series: it shows that the hole at the origin, observed on any finite portion of the orbit if $0.5 < \sigma < 1$, quickly vanishes if you slightly modify the series.

Here I focus on randomizing $L_P(z, \chi)$. Let $z = \sigma + it$ be fixed, $0.5 < \sigma < 1$, and for each prime $p \in P$, $\chi(p)$ be a random variable equal to $+1$ or -1 with probability 0.5. The $\chi(p)$'s are assumed to be independent. It follows immediately that

$$\mathrm{E}[L_P(z, \chi)] = \prod_{p \in P} \mathrm{E}\Big[\frac{1}{1 - \chi(p) p^{-z}}\Big] = \prod_{p \in P} \frac{1}{1 - p^{-2z}} = L_P(2z, \chi_0),$$

where $\chi_0(\cdot)$ is the constant function equal to one. Note that the product converges if $\sigma > 0.5$. So this type of randomization extents the abscissa of convergence from $\sigma > 1$ to $\sigma > 0.5$. Now let the random variable ρ^2 be the square of the distance to the origin:

$$\rho^2 = |L_P(z, \chi)|^2 = \prod_{p \in P} |1 - \chi(p) p^{-z}|^{-2} = \prod_{p \in P} \frac{1}{1 - 2p^{-\sigma} \chi(p) \cos(t \log p) + p^{-2\sigma}}. \tag{14}$$

We have

$$\mathrm{E}[\rho^2] = \prod_{p \in P} \mathrm{E}\Big[|1 - \chi(p) p^{-z}|^{-2}\Big] = \prod_{p \in P} \frac{1 + p^{-2\sigma}}{(1 + p^{-2\sigma})^2 - 4p^{-2\sigma} \cos^2(t \log p)}.$$

In particular,

$$\prod_{p \in P} \frac{1}{1 + p^{-2\sigma}} = \frac{L_P(4\sigma, \chi_0)}{L_P(2\sigma, \chi_0)} \leq \mathrm{E}[\rho^2] \leq \prod_{p \in P} \frac{1 + p^{-2\sigma}}{(1 - p^{-2\sigma})^2} = \frac{L_P^3(2\sigma, \chi_0)}{L_P(4\sigma, \chi_0)}. \tag{15}$$

Again, $\chi_0(\cdot)$ is the constant function equal to one. The maximum is attained when $t = 0$. If P is the full set of primes, Formula (15) becomes $\zeta(4\sigma)/\zeta(2\sigma) \leq \mathrm{E}[\rho^2] \leq \zeta^3(2\sigma)/\zeta(4\sigma)$, where ζ is the Riemann zeta function

[Wiki]. Similar bounds are available for $\mathrm{E}[\rho]$. It is interesting to note that by averaging over all potential $\chi(\cdot)$'s, the orbit has a hole encompassing the origin, with a strictly positive radius. This is in contrast to the non-random case, where the hole is reduced to a point regardless of χ (assuming $0.5 < \sigma < 1$).

Let $P = \{p_1, p_2, \cdots\}$ with the primes listed in increasing order. Now let us define the random variable $L_\chi(n) = \sum_{k=1}^{n} \chi(p_k)$. It satisfies the law of the iterated logarithm [Wiki], stated in Formula (7), and translating here to

$$\limsup_{n\to\infty} \frac{|L_\chi(n)|}{\sqrt{n \log\log n}} = C,$$

for some constant C with $0 < C < \infty$. But the prime numbers are not perfectly random, and one would expect $\sqrt{n \log\log n}$ to be replaced by (say) $\sqrt{n^{1+\epsilon}}$ for any arbitrary small $\epsilon > 0$. This may be the case in the deterministic example where $\chi(p) = +1$ if $p \equiv 1 \bmod 4$ and $\chi(p) = -1$ if $p \equiv 3 \bmod 4$. One way to make the random Euler product more realistic is to introduce weak dependencies among the $\chi(p)$'s. Then one can test whether the improved stochastic model (with weak dependencies) is a better fit to the observed data – the actual $\chi(p)$'s. The weak dependencies can be introduced as follows, when simulating $\chi(p_{n+1})$ for n large enough:

- If $|L_\chi(n)| < \beta\sqrt{n(\log\log\log n)^\nu}$, then $P[\chi(p_{n+1}) = 1] = \frac{1}{2} + \mu n^{-\alpha}$, $P[\chi(p_{n+1}) = -1] = \frac{1}{2} - \mu n^{-\alpha}$.
- Otherwise choose between $+1$ and -1 so that $|L_\chi(n+1)| < |L_\chi(n)|$.

You may try with various values of $\alpha, \beta, \nu > 0$ and μ to see which ones provide the best fit for very large n, say $n > 10^{15}$. You could also change the sign of μ every now and then. The choice of $\log\log\log n$ is inspired by Gonek's conjecture: see [17] page 29.

Finally, another random Euler product also investigated in [18] is the following:

$$\rho^2 = \prod_{p\in P} \frac{1}{1 - 2p^{-\sigma}\cos(\Theta_p) + p^{-2\sigma}}. \tag{16}$$

Here the Θ_p's are independent uniform deviates on $[0, 2\pi]$. Compare this to formula (14). Since

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - 2p^{-\sigma}\cos\theta + p^{-2\sigma}} d\theta = \frac{1}{1 - p^{-2\sigma}},$$

we have

$$\mathrm{E}[\rho^2] = \prod_{p\in P} \mathrm{E}\left[\frac{1}{1 - 2p^{-\sigma}\cos(\Theta_p) + p^{-2\sigma}}\right] = \prod_{p\in P} \frac{1}{1 - p^{-2\sigma}} = L_P(2\sigma, \chi_0),$$

where again, $\chi_0(\cdot)$ is the constant function equal to one. If P is the full set of primes, $L_P(2\sigma, \chi_0) = \zeta(2\sigma)$. The proof that the distribution attached to ρ^2 exists and is not singular, can be found in [18].

4 Exercises

The following exercises require out-of-the-box thinking. They complement the theory or provide a proof to some of the theoretical results discussed in this paper.

Exercise 1 – Asymptotic formula. Prove the asymptotic formula (2).

Solution

By definition, we have $q_k = p_1^{a_1} p_2^{a_2} \cdots p_d^{a_d}$ for some positive integers $a_1, \dots, a_d$. In other words, $a_1 \log p_1 + \cdots + a_d \log p_d = \log q_k$. This is the equation of a $d-1$ dimensional simplex with vertices $(\log p_1, 0, \dots, 0)$, $(0, \log p_2, \dots, 0)$ ... $(0, 0, \dots, \log p_d)$. If you add the origin as a vertex, then the number of points v_k with integer coordinates, inside the newly created d-dimensional simplex, is approximately equal to the volume V_k of that simplex [Wiki] . Also, v_k is the number of positive integers less than or equal to q_k, since each integer has a unique factorization in (ordered) prime numbers. So, q_k is the inverse of the function v_k, which is asymptotically equal to the inverse of V_k. To complete the proof, use the well known fact that

$$V_k = \frac{1}{d!}\Big(\log q_k\Big)^d \prod_{p\in P} \frac{1}{\log p} = \frac{1}{d!}\prod_{p\in P} \log_p q_k,$$

where $\log_p$ stands for the logarithm in base p. The result is easy to verify if $d = 1$.

Exercise 2 – **Equivalence between series and Euler product**. Prove formula (1).

Solution

Expanding the product in formula (1), one obtains

$$\prod_{p\in P}\frac{1}{1-p^{-z}}=\prod_{p\in P}\Big(1+p^{-z}+p^{-2z}+\cdots\Big)=\sum_{a_1,a_2,\dots,a_d}\Big(p_1^{a_1}p_2^{a_2}\cdots p_d^{a_d}\Big)^{-z}=\sum_{k=1}^{\infty}q_k^{-z}.$$

Let us denote the rightmost series in the above equation a $\zeta_P(z)$. To complete the proof, use the fact that

$$\sum_{k=1}^{\infty}\delta_k q_k^{-z}=\zeta_P(z)-2\sum_{q_k\text{ even}}q_k^{-z}=\zeta_P(z)-2\sum_{k=1}^{\infty}(2q_k)^{-z}=(1-2^{1-z})\zeta_P(z).$$

Another interesting identity is the following one:

$$(1-2^{1-z})\prod_{p\in P}\frac{1}{1+p^{-z}}=(1-2^{1-z})\frac{\zeta_P(2z)}{\zeta_p(z)}=\sum_{k=1}^{\infty}\delta_k\lambda(q_k)q_k^{-z},$$

where $\lambda(\cdot)$ is the Liouville function [Wiki].

Exercise 3 – **Convergence problem**. If P is the set of all primes and δ_k is replaced by $+1$, then the series in formula (1) will not converge if $\sigma<1$. Here, $z=\sigma+it$. That is, σ is the real part of the complex number z.

Solution

In this case $q_k=k$. Note that $k^{-z}=k^{-\sigma}\cdot[\cos(t\log k)+i\sin(t\log k)]$. It suffices to prove that the series with k-th term equal to $k^{-\sigma}\cos(t\log k)$ can not converge, even though $\cos(t\log k)$ oscillates infinitely often between positive and negative values, as k increases. The reason is because $\log k$ grows too slowly. When k is very close to a multiple of (say) 2π, too many consecutive terms are all positive, and k^{σ} does not grow fast enough (if $\sigma<1$) to keep the partial sums bounded and converging. To the contrary, if $\log k$ is replaced by $\sqrt{k}$ in the cosine function, and $\sigma=0.75$, then the series converge: the corresponding integral between 0 and ∞ is equal to $\sqrt{2\pi/|t|}$.

Exercise 4 – **Truncated Dirichlet function**. Determine the area covered by the orbit, for a fixed value of σ, when $n=3$ and t runs though all the positive real numbers in Formula (8). Generalize to $n=4$.

Solution

Let us assume that the center of the hole is $(1,0)$ in this case. Proving it is left to the reader. Then the square of the distance between a point on the orbit at time t, and the origin, is equal to

$$\begin{aligned}d^2(t)&=\Big[\beta^{\sigma}\cos(t\log 2)-\lambda_3^{\sigma}\cos(t\log 2)\Big]^2+\Big[\beta^{\sigma}\sin(t\log 2)-\lambda_3^{\sigma}\sin(t\log 2)\Big]^2\\&=\beta^{2\sigma}+\lambda_3^{2\sigma}-2\beta^{\sigma}\lambda_3^{\sigma}\Big[\cos(t\log 2)\cos(t\log 3)+\sin(t\log 2)\sin(t\log 3)\Big]\\&=\beta^{2\sigma}+\lambda_3^{2\sigma}-2\beta^{\sigma}\lambda_3^{\sigma}\cos\Big(t\log\frac{2}{3}\Big)\end{aligned}$$

Since the cosine takes all values between -1 and $+1$, infinitely often with period $2\pi/(\log 3-\log 2)$, the outer boundary of the orbit is a circle of radius ρ_1 centered at $(1,0)$, and the inner boundary (that is, the shape of the hole) is a circle of radius ρ_2 also centered at $(1,0)$. Here

$$\rho_1=\sqrt{\beta^{2\sigma}+\lambda_3^{2\sigma}+2\beta^{\sigma}\lambda_3^{\sigma}}=|\beta^{\sigma}+\lambda_3^{\sigma}|,\quad \rho_2=\sqrt{\beta^{2\sigma}+\lambda_3^{2\sigma}-2\beta^{\sigma}\lambda_3^{\sigma}}=|\beta^{\sigma}-\lambda_3^{\sigma}|.$$

Thus the larger β, the bigger the hole. I assumed that $\beta,\lambda_3\geq 0$.

The case $n=4$ is considerably more complicated. The center is no longer $(1,0)$, but typically $(c,0)$ with $c<1$. Also, the shapes may no longer be circles. There may or may not be a hole. However, if β is large enough, there will be a hole, big enough to encompass $(1,0)$. The square of the distance to $(1,0)$ is now

$$d^2(t)=\beta^{2\sigma}+\lambda_3^{2\sigma}+\lambda_4^{2\sigma}-2\beta^{\sigma}\lambda_3^{\sigma}\cos\Big(t\log\frac{2}{3}\Big)-2\beta^{\sigma}\lambda_4^{\sigma}\cos\Big(t\log\frac{2}{4}\Big)-2\lambda_3^{\sigma}\lambda_3^{\sigma}\cos\Big(t\log\frac{3}{4}\Big).$$

The minimum possible value for $d^2(t)$ is $d_{\min}=\beta^{2\sigma}+\lambda_3^{2\sigma}+\lambda_4^{2\sigma}-2\beta^{\sigma}\lambda_3^{\sigma}-2\beta^{\sigma}\lambda_4^{\sigma}-2\lambda_3^{\sigma}\lambda_3^{\sigma}$. If $d_{\min}>0$, there is a hole big enough to encompass $(1,0)$.

Exercise 5 – The orbit covers a dense area in the complex plane. The purpose of this exercise is to prove a particular case: if $\sigma < 1$, then $\liminf |\text{Ł}_P(z,\chi)| = 0$ regardless of χ. The infimum [Wiki] is over all complex numbers z. Here we assume that P is an infinite subset of primes (or the full set), and that the sum of $p^{-\sigma}$ over all $p \in P$, diverges. This proves that the orbit is dense around the origin under certain conditions. It implies, under these conditions, that the hole shrinks to a singleton (the origin) if $|L_P(z,\chi)| > 0$ for all z, and to an empty set if $L_P(z,\chi) = 0$ for some z.

Solution

Assume the Euler product is finite, and contains only the first d primes $p_1, \dots, p_d \in P$. Let $z = \sigma + it$ as usual, with t large enough so that $t \log p_k$ gets extremely close to a multiple of π, say $m_k\pi$, for all $k = 1, \dots, d$. This is possible thanks to Kronecker's theorem [Wiki]. Then $\sin(t \log p_k)$ gets very close to 0, and $\cos(t \log p_k)$ gets very close to either -1 or $+1$ depending on whether m_k is odd or even. Thus, the imaginary part of $L_P(z,\chi)$, involving the sine terms only, gets very close to 0. The real part, involving the cosine terms only, gets very close to

$$S(\chi^\star) = \sum_{k=1}^{\infty} \chi^\star(k)\chi(k)k^{-\sigma},$$

where

- The function $\chi^\star$ is a completely multiplicative [Wiki] and thus entirely defined by its values on prime numbers,
- $\chi^\star(p_k) \in \{1, +1\}$ if $p_k \in P$ and $1 \leq k \leq d$, otherwise $\chi^\star(p_k) = 0$,
- $\chi^\star(p_k) = +1$ if m_k is even, otherwise $\chi^\star(p_k) = -1$.

Also, $p^{-z} = [\cos(t \log p_k) + i \sin(t \log p_k)] \cdot p^{-\sigma} \to \chi^\star(p_k)p^{-\sigma}$ as $t \to \infty$. Thus for the Euler product, with the special t discussed above (a function of $m_1, \dots, m_d$) and $z = \sigma + it$, we have

$$L_P(z,\chi) \to \prod_{k=1}^{d} \frac{1}{1 - \chi^\star(p_k)\chi(p_k)p_k^{-\sigma}} \in \mathbb{R} \text{ as } t \to \infty. \tag{17}$$

If $\sum_{p \in P} p^{-\sigma} = \infty$, then the product in Formula (17) can approximate any positive value arbitrarily closely, as $d \to \infty$. Indeed, $d_k = \chi^\star(p_k)$ can be seen as the k-th binary digit of the (real) number $L_P(z,\chi)$ in some special numeration system. And you can compute d_k using a technique similar to that used for standard binary digits in base 2, with a version of the greedy algorithm [Wiki]. To get arbitrarily close to zero, one way is to choose the function $\chi^\star$ so that $\chi^\star\chi = -1$.

Note: In the computation of $S(\chi^\star)$, I implicitly used the following facts. First, if k has the prime factorization $k = p_1^{a_1} \cdots p_d^{a_d}$, then $\cos(t \log k) = \cos(a_1 t \log p_1 + \cdots + a_d t \log p_d)$. Recursively using $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, with the fact that all the sines are zero and $\cos(a_i t \log p_i) \to \cos(a_i m_i \pi) = [\chi^\star(p_i)]^{a_i}$ as $t \to \infty$, one obtains $\cos(t \log k) \to [\chi^\star(p_1)]^{a_1} \cdots [\chi^\star(p_d)]^{a_d} = \chi^\star(k)$.

Exercise 6 – Density of integers of the form $x^3 + y^2$. Prove that k-th integer of the form $x^3 + y^2$ is asymptotically larger than $k^{6/5}$.

Solution

Let $v(n)$ be the number of lattice points (x, y), with x, y positive integers, satisfying $x^3 + y^2 \leq n$. Estimating v_n is a classic problem in additive number theory [Wiki], generalizing the Gaussian circle problem [Wiki]. The solution to this class of problems is as follows. Let $S_1, \dots, S_m$ be m infinite sets of positive integers, and $v_i(n)$ be the number of elements less than n in S_i. Let $v(n)$ be the number of lattice points $(x_1, \cdots, x_m)$, with $x_i \in S_i$, satisfying $x_1 + \cdots + x_m \leq v(n)$. If $v_i(n) \sim a_i n^{b_i} (\log n)^{-c_i}$ with $0 < b_i \leq 1, c_i \geq 0$ and $a_i > 0$ $(i = 1, \cdots m)$, then

$$v(n) \lesssim \frac{\prod_{i=1}^{m} a_i \Gamma(b_i + 1)}{\Gamma(1 + \sum_{i=1}^{m} b_i)} \cdot \frac{n^{b_1 + \cdots + b_m}}{(\log n)^{c_1 + \cdots + c_m}},$$

where $\lesssim$ means "asymptotically smaller", and Γ is the Gamma function. See here and here for details.

In our case, $m = 2, a_1 = a_2 = 1, c_1 = c_2 = 0, b_1 = 1/3, b_2 = 1/2$. Thus $v(n) \lesssim n^{5/6}$. Note that this method may result in double counting, for instance $225 = 6^3 + 3^2 = 5^3 + 10^2 = 0^3 + 15^2$. Thus the actual number $w(n)$ of positive integers of the form $x^3 + y^2$ is even smaller, thus definitely smaller than $n^{5/6}$. It follows immediately, using the inverse of the function $w(n)$, that the k-th element is asymptotically larger than $n^{6/5}$.

Exercise 7 – Strange factorization of the Dirichlet functions. Show, based on the Euler product, that $L_P(z,\chi) = L_P(z/2,\psi)L_P(z/2,-\psi)$ where $\psi^2(p) = \chi(p)$, thus $\psi(p) \in \{1, -1, i, -i\}$. In particular, if $\chi(p) = 1$, one can choose $\psi(p_{2k+1}) = 1$ and $\psi(p_{2k}) = -1$ where p_k is the k-th prime. In this case, $L(z,\chi) = \zeta(z)$, and the

Euler products of $L(z/2, \psi)$ and $L(z/2, -\psi)$ both converge if $\sigma > 0$. However, $L_P(z/2, \psi)L_P(z/2, -\psi) = \zeta(z)$ only if $\sigma > 1$. How can these formulas be generalized recursively?

Solution

Let $\psi_0 = \chi$ and $\psi_1 = \psi$. Using the same logic, $L(z/2, -\psi_1) = L_P(z/4, \psi_2)L_P(z/4, -\psi_2)$ where $\psi_2^2(p) = -\psi_1(p)$. Applying this method recursively, one obtains

$$L_P(z, \chi) = L(z/2^n, -\psi_n) \prod_{k=1}^{n} L_P(z/2^k, \psi_k),$$

where $n \geq 1$ and $\psi_{k+1}^2 = -\psi_k$ for $k = 1, 2$ and so on. For which values of σ is this formula valid? Based on the construction, all the associated Euler products must converge, suggesting $\sigma > 2^n$.

Exercise 8 – Roots of the Riemann zeta function. As usual, $z = \sigma + it$. Let S_0 be any open interval containing exactly one value t_0 such that $\zeta(\frac{1}{2} + it_0) = 0$, and let $\eta(z)$ be the standard Dirichlet eta function. Discuss the existence (or not) of roots of $\eta(z)$ if $t \in S_0$ and $\frac{1}{2} < \sigma < 1$. Show how it works when $S_0 =]199, 202[$. You can find a table of the first 100,000 non-trivial roots of $\zeta(z)$, here.

Solution

Assume σ is fixed. Let $\mu(\sigma)$ be the minimum of $|\eta(z)|$ if $t \in S_0$, and let $\tau(\sigma)$ be the value achieving the minimum. That is,

$$\mu(\sigma) = \min_{t \in S_0} |\eta(\sigma + it)|, \quad \tau(\sigma) = \arg\min_{t \in S_0} |\eta(\sigma + it)|.$$

If $S_0 =]199, 202[$ then $t_0 \approx 201.26$. Let $t_0' = 44\pi / \log 2 \approx 199.42$. We have $1 - 2^{1-z} = 0$ where $z = 1 + it_0'$, and:

- $\mu(\frac{1}{2}) = 0$ and $\tau(\frac{1}{2}) = t_0$ since $\eta(\frac{1}{2} + it_0) = 0$,
- $\mu(1) = 0$ and $\tau(1) = t_0'$, since $\eta(1 + it_0') = 0$,
- $\mu(\sigma)$ is strictly increasing and continuous if $\frac{1}{2} \leq \sigma \leq \sigma_0$ with $\sigma_0 \approx 0.75$,
- $\mu(\sigma)$ is strictly decreasing and continuous if $\sigma_0 \leq \sigma \leq 1$.

In other words, $\mu(\sigma)$ is convex. It seems to imply that there is no root if $t \in S_0$ and $\frac{1}{2} < \sigma < 1$. However, proving that $\mu(\sigma)$ is increasing or decreasing, with only one change-point at some σ_0 (depending on S_0), may be as hard as proving RH itself: it is based on empirical evidence only, and related to the (conjectured) absence of roots for the derivative of $\zeta(z)$. Indeed, it is just a consequence of the Riemann Hypothesis, see here.

There is something particularly striking though, which could prove useful to make some progress: the function $\tau(\sigma)$ is almost flat, with a single discontinuity at σ_0. Indeed, $201.26 \leq \tau(\sigma) \leq 201.29$ if $\frac{1}{2} \leq \sigma < \sigma_0$, and $199.41 \leq \tau(\sigma) \leq 199.42$ if $\sigma_0 < \sigma \leq 1$. These variations are so small that you wonder if they are real, or caused by numerical imprecision (implying the function $\tau(\sigma)$ could be perfectly flat with a single discontinuity, making it potentially easier to prove RH). If these variations really do exist, there might be a function other than $\eta(z)$ with the same roots, say a scaled version of $\eta(z)$ with a scaling factor free of roots, for which the variations are absent. Such a function may be easier to investigate.

More generally, the functions $\mu(\sigma)$ and $\tau(\sigma)$ have this same behavior whenever S_0 contains a value t_0' such that $z = 1 + it_0'$ is a root of $1 - 2^{1-z}$, and therefore a root of $\eta(z)$. Because these roots are evenly spaced by the increment $2\pi / \log 2$, and since the roots at $\sigma = \frac{1}{2}$ are closer and closer to each other as $t \to \infty$ (see [3, 20]), there is either one t_0' in S_0, or none. There can't be more than one. When there is none, the situation is even easier and amounts to setting $\sigma_0 = \infty$, or at least $\sigma_0 > 1$.

Finally, if it was possible to prove the points discussed in this exercise, then of course RH would be proved. It suffices to consider the collection of all possible S_0 to show that there would be no root anywhere, no matter how large t is, if $\frac{1}{2} < \sigma < 1$.

Exercise 9 – Approximating the Dirichlet eta function. The Dirichlet series for $\eta(z)$ converges very slowly and chaotically, especially if σ is small or t is large. One way to accelerate the convergence is to use Euler's transform [Wiki]. See also Exercise 25 in [11]. Other approximations exist, for instance using Dirichlet polynomials [7]. Here I investigate yet a different type of approximation. Let $z = \sigma + it$ as usual, with σ fixed, say $\sigma = 0.8$. Also assume that the values of $\eta(\sigma + ik)$ are known and denoted as $\eta_k(\sigma)$ if k is a positive integer. The approximation is as follows:

$$\eta(z) \approx \frac{\sin \pi t}{\pi} \cdot \left[\frac{\eta_0(\sigma)}{t} + 2t \sum_{k=1}^{n} (-1)^k \frac{\eta_k(\sigma)}{t^2 - k^2} \right]$$

Show how good this approximation is. For the solution, see Exercise 2 in [10].

5 Python code

The main code is in section 5.1. The code in section 5.2 is provided for convenience only: it does not further illustrate the theory, but instead focuses on producing beautiful videos of the orbits studied in this paper. My article on the prime test for pseudo-random number generators [9], deeply related to the Generalized Riemann Hypothesis, has more Python code directly relevant to the topics discussed here.

5.1 Computing the orbit of various Dirichlet series

The code below computes $\eta_P(z, \chi)$ if the `Dirichlet` variable is set to `True`; otherwise it computes $L_P(z, \chi)$. More specifically, it computes the value of the function in question for $z = \sigma + it$, with $\sigma = \Re(z) \geq 0.5$ fixed and determined by the variable `sig`, and for equally spaced values of t between `minT` and `maxT`. The spacing is determined by the variable `increment`, typically set to 0.01. It uses the Dirichlet series expansion, with the number of tems determined by `nterms` (typically set to $2,000$) to get at least 2 digits of accuracy in the worse case where $\sigma = 0.5$ or t is large. The function `primes.check()` from the primePy library tests if a number is a prime. The code actually handles the more general case where $\chi(p)$ is replaced by $\chi(p) \cdot x^{\nu(p)}$, with $\nu(p) = 1/p$ as in Formula (12) and $0 < x \leq 1$. The standard case corresponds to $x = 1$. The variable x is represented by `x` in the code.

The value of $\eta_P(z, \chi)$ or $L_P(z, \chi)$ is a complex number: its real and imaginary parts are respectively named `etax` and `etay` in the code. The function χ, and thus the set P, depends on the option selected in the code, determined by the variable `method`: `Zeta` (that is, $\zeta(z)$ by default), `Eta`, `Dirichlet4` (corresponding to χ_4), `Beurling`, `Alternating`, or `Random`. Please refer to the text to identify when the series converge or not: in particular, the `Zeta` method converges only if $\sigma > 1$.

The output variables `minL` and `maxL` help determine convergence status. They are discussed in more details in my article on pseudo-random number generators [9]. The parameter `beta` should be set to 0.5, unless you want to replicate the experiments discussed in section 3.1, where `beta` is represented by β. Likewise, keep `x` set to 1, unless you want to replicate the experiments in section 3.2.4. Finally, the program uses hash tables (dictionaries in Python) rather than arrays, for increased efficiency: these arrays would be quite sparse. The source code (below) is also on GitHub: look for `DirichletL.py`.

```python
# DirichletL.py. Generate orbits of various Dirichlet-L and related functions
# By Vincent Granville, https://www.MLTechniques.com/resources/

import math
import random
from primePy import primes

nterms=2000 # increase to 10000 for sig = 0.5
method='Eta'
sig=0.9
Dirichlet=False
x=1         # must have 0 < x <= 1; default is x=1
beta=0.5    # beta > 0.5 magnifies the hole of the orbit

random.seed(1)
primeSign={}
start=2
if method=='Dirichlet4':
  start=3
idx=0
for k in range(start,nterms):
  if primes.check(k):
    idx=idx+1
    p=k
    xpow=x**(1/p)
    if method=='Beurling' and p==3:
      p=2+math.log(3)
    primeSign[p]=xpow
    if method=='Dirichlet4' and k%4==3:
      primeSign[p]=-xpow
    elif method=='Alternating' and idx%2==1:
      primeSign[p]=-xpow
```

```
   elif method=='Random' and random.random()>0.5:
     primeSign[p]=-xpow
   elif method=='Eta':
     Dirichlet=True

signHash={}
evenHash={}
signHash[1]=1
evenHash[1]=0      # largest power of 2 dividing k
for p in primeSign:
  if p*math.pi %1 < 0.05:
    print(p,"/",nterms) # show progress (where we are in the loop)
  oldSignHash={}
  for k in signHash:
    oldSignHash[k]=signHash[k]
  for k in oldSignHash:
    pp=1
    power=0
    localProduct=oldSignHash[k]
    while k*p*pp<nterms:
      pp=p*pp
      power=power+1
      new_k=k*pp
      localProduct=localProduct*primeSign[p]
      signHash[new_k]=localProduct
      if p==2:
        evenHash[new_k]=power
      else:
        evenHash[new_k]=evenHash[k]

for k in sorted(evenHash):
  if Dirichlet and evenHash[k]>0:
    signHash[k]=-signHash[k]

sumL=0
minL= 2*nterms
maxL=-2*nterms
argMin=-1
argMax=-1
denum={}
tlog={}
for k in sorted(signHash):
  denum[k]=signHash[k]/k**sig
  tlog[k]=math.log(k)
  sumL=sumL+signHash[k]
  if sumL<minL:
    minL=sumL
    argMin=k
  if sumL>maxL:
    maxL=sumL
    argMax=k
denum[2]=signHash[2]/(1/beta)**sig

def G(tau,sig,nterms):
  fetax=0
  fetay=0
  for j in sorted(signHash):
    fetax=fetax+math.cos(tau*tlog[j])*denum[j]
    fetay=fetay+math.sin(tau*tlog[j])*denum[j]
  return [fetax,fetay]

minT=0.0
maxT=2000.0
increment=0.05

OUT = open("dirichletL.txt", "w")
```

```
t=minT
loop=0
while t <maxT:
  if loop%100==0:
    print("t= %5.2f / %d" % (t,maxT))
  loop=loop+1
  (etax,etay)=G(t,sig,nterms)
  line=str(t)+"\t"+str(etax)+"\t"+str(etay)+"\n"
  OUT.write(line)
  t=t+increment
OUT.close()

print("\n")
print(argMin,"-->",minL)
print(argMax,"-->",maxL)
```

5.2 Creating videos of the orbit

The Python code in this section deals with the visualization aspects: producing data animations (MP4 videos) of three orbits of $\eta_P(z, \chi)$, when P is the full set of prime integers and $\chi(\cdot)$ is the contant function equal to 1. This is the the standard Dirichlet function. The three orbits in question correspond to $\sigma = 0.5$, $\sigma = 0.75$ and $\sigma = 1.25$. These three values are set by the instructions `sigma.append()` in the code. In particular, $\sigma = 0.5$ reveals the infinitely many roots, while the two other values show the lack of root.

The output videos are available on my GitHub repository, here. The videos are also on YouTube, here. For convenience, the Python code is also included in this article. Top variables include `ShowOrbit` (set to `True` if you want to display the orbit, not just the points), `dot` (the size of the dots), `r` (when iterating over time, it outputs a video frame once every r iterations), `width` and `height` (the dimension of the image). The final image is eventually reduced by half due to the anti-aliasing procedure used to depixelate the curves. This is performed within `img.resize` in the code, using the `Image.LANCZOS` parameter [Wiki]. Segments joining two dots on the orbit (to create the appearance of a smooth, curvy orbit) are produced using the Pillow library and its `ImageDraw` functions.

Reducing the size of the image and the number of frames per second (FPS) will optimize speed and disk usage. The biggest improvement, in terms of speed, is replacing all numpy calls (`np.log`, `np.cos` and so on) by math calls (`math.log`, `math.cos` and so on). If you use numpy for image production rather than Pillow, the opposite may be true (I did not test). The source code is also on GitHub: look for `image3R_orbit_enhanced.py`.

```
# image3R_orbit_enhanced.py [www.MLTechniques.com]

from PIL import Image, ImageDraw # ImageDraw to draw ellipses etc.
import moviepy.video.io.ImageSequenceClip # to produce mp4 video
from moviepy.editor import VideoFileClip # to convert mp4 to gif

import numpy as np
import math
import random
random.seed(100)

#--- Global variables ---

m=3            # number of orbits (one for each value of sigma)
nframe=10000   # number of images created in memory
ShowOrbit=True
ShowDots=False
count=0        # frame counter
r=10           # one out of every r image is included in the video
dot=4          # size of a point in the picture
step=0.01      # time increment in orbit

width = 3200  # width of the image
height =2400  # length of the image
```

```
images=[]

etax=[] # real part of Dirichlet eta function
etay=[] # real part of Dirichlet eta function
sigma=[] # imaginary part of argument of Dirchlet eta
x0=[]    # value of etax on last video frame
y0=[]    # value of etay on last video frame
#col=[] # RGB color of the orbit
colp=[] # RGP points on the orbit
t=[]     # real part of argument of Dirchlet eta (that is, time in orbit)
flist=[] # filenames of the images representing each video frame

etax=list(map(float,etax))
etay=list(map(float,etay))
sigma=list(map(float,sigma))
x0=list(map(float,x0))
y0=list(map(float,y0))
t=list(map(float,t))
flist=list(map(str,flist))

#--- Eta function ---

def G(tau,sig,nterms):
  sign=1
  fetax=0
  fetay=0
  for j in range(1,nterms):
    fetax=fetax+sign*math.cos(tau*math.log(j))/pow(j,sig)
    fetay=fetay+sign*math.sin(tau*math.log(j))/pow(j,sig)
    sign=-sign
  return [fetax,fetay]

#--- Initializing comet parameters ---

for n in range (0,m):
  etax.append(1.0)
  etay.append(0.0)
  x0.append(1.0)
  y0.append(0.0)
  t.append(0.0)   # start with t=0.0
sigma.append(0.50)
sigma.append(0.75)
sigma.append(1.25)
colp.append((255,0,0,255))
colp.append((0,0,255,255))
colp.append((255,180,0,255))

if ShowOrbit:
  minx=-2
  maxx=3
else:
  minx=-1
  maxx=2

rangex=maxx-minx
rangey=0.75*rangex
miny=-rangey/2
maxy=rangey/2
rangey=maxy-miny

img = Image.new( mode = "RGB", size = (width, height), color = (255, 255, 255) )
imgCopy=img.copy()
draw = ImageDraw.Draw(img,"RGBA")
drawCopy = ImageDraw.Draw(imgCopy,"RGBA")

gx=width*(0.0-minx)/rangex
```

```
gy=height*(0.0-miny)/rangey
hx=width*(1.0-minx)/rangex
hy=height*(0.0-miny)/rangey
draw.ellipse((gx-8, gy-8, gx+8, gy+8), fill=(0,0,0,255))
draw.ellipse((hx-8, hy-8, hx+8, hy+8), fill=(0,0,0,255))
draw.rectangle((0,0,width-1,height-1), outline ="black",width=1)
draw.line((0,gy,width-1,hy), fill ="red", width = 1)
draw.ellipse((gx-8, gy-8, gx+8, gy+8), fill=(0,0,0,255))
drawCopy.ellipse((hx-8, hy-8, hx+8, hy+8), fill=(0,0,0,255))
drawCopy.rectangle((0,0,width-1,height-1), outline ="black",width=1)
drawCopy.line((0,gy,width-1,hy), fill ="red", width = 1)
countCopy=0

#--- Main Loop ---

for k in range (2,nframe,1): # loop over time, each t corresponds to an image
  if k %10 == 0:
    string="Building frame:" + str(k) + "> "
    for n in range (0,m):
      string=string+ " | " + str(t[n])
    print(string)
  if k%r==0:
    imgCopy.paste(img, (0, 0))
  for n in range (0,m): # loop over the m orbits
    if ShowOrbit:
      # save old value of etax[n], etay[n]
      x0.insert(n,width*(etax[n]-minx)/rangex)
      y0.insert(n,height*(etay[n]-miny)/rangey)
    (etax[n],etay[n])=G(t[n],sigma[n],2000)
    x= width*(etax[n]-minx)/rangex
    y=height*(etay[n]-miny)/rangey
    if ShowOrbit:
      if k>2:
        # draw line from (x0[n],y0[n]) to (x,y)
        draw.line((int(x0[n]),int(y0[n]),int(x),int(y)), fill =colp[n], width = 0)
        if ShowDots:
          draw.ellipse((x-dot, y-dot, x+dot, y+dot), fill =colp[n])
        else:
          copyFlag=True
          drawCopy.ellipse((x-10, y-10, x+10, y+10), fill =colp[n])
      t[n]=t[n]+step
    else:
      draw.ellipse((x-dot, y-dot, x+dot, y+dot), fill =colp[n])
      t[n]=t[n]+200*math.exp(3*sigma[n])/(1+t[n]) # 0.02
  if k%r==0: # this image gets included as a frame in the video
    draw.ellipse((gx-8, gy-8, gx+8, gy+8), fill=(0,0,0,255))
    draw.ellipse((hx-8, hy-8, hx+8, hy+8), fill=(0,0,0,255))
    drawCopy.ellipse((gx-8, gy-8, gx+8, gy+8), fill=(0,0,0,255))
    drawCopy.ellipse((hx-8, hy-8, hx+8, hy+8), fill=(0,0,0,255))
    fname='imgpy'+str(count)+'.png'
    count=count+1
    # anti-aliasing mechanism
    if not copyFlag:
      img2 = img.resize((width // 2, height // 2), Image.LANCZOS) #ANTIALIAS)
    else:
      img2 = imgCopy.resize((width // 2, height // 2), Image.LANCZOS) #ANTIALIAS)
    # output curent frame to a png file
    img2.save(fname) # write png image on disk
    flist.append(fname) # add its filename (fname) to flist

# output video file
clip = moviepy.video.io.ImageSequenceClip.ImageSequenceClip(flist, fps=20)
clip.write_videofile('riemann.mp4')
```

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